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STUDY ON FORCED CONVECTIVE HEAT TRANSFER IN CURVED PIPES

(3RD REPORT, THEORETICAL ANALYSIS UNDER THE CONDITION OF UNIFORM WALL TEMPERATURE AND PRACTICAL FORMULAE)

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Abstract—In the authors' previous reports, the theoretical and experimental studies on flow and temperature fields in a curved pipe were made under the condition of uniform heat flux. In the former part of the present report, a theoretical analysis is made about temperature field far downstream from the pipe inlet under the condition of uniform wall temperature, following the same procedure as in the previous papers. Nusselt number is found to be remarkably affected by a secondary flow due to curvature. The result shows that in the first-order approximation the Nusselt number of heat transfer in a curved pipe does not differ for uniform wall temperature or uniform heat flux case, in both laminar and turbulent regions.

In the latter part, the formulae of Nusselt numbers obtained by the authors' study are arranged so as to have simpler expression for a practical use.

It is also investigated what temperature should be chosen in calculation of physical properties when these Nusselt number formulae are used.

NOMENCLATURE

A, w_1 at the center of a cross section perpendicular to a pipe axis;

a, radius of a pipe;

C, coefficient, $-(\partial P/\partial z)$;

C_M, coefficient giving T_m [equation (60)];

c_p, specific heat of fluid at constant pressure;

D, dimensionless velocity of the secondary flow in a flow core;

f_i, eigenfunction of $(T_w - T)$;

f_0, eigenfunction for α'_0;

g, dimensionless temperature $(T_w - T)/(T_w - T_m)$;

h, heat-transfer coefficient, $Nu_c k/2a$;

h_m, mean heat-transfer coefficient between T_0 and T_1;

K, Dean number, $\equiv Re\sqrt{(a/R)}$;

k, heat conductivity;

N, normalizing constant for g [equation (21)];

Nu, Nusselt number, $\equiv [2aQ_{wm}/k(T_w - T_m)]$;

P, dimensionless pressure, $\equiv (a^2/\nu^2)(p/\rho)$;

Pr, Prandtl number, $\equiv \mu c_p/k$;

p, pressure;

Q, heat flux;

Q_w, heat flux at the wall;

Q_{wm}, mean value of Q_w around the periphery ($\psi = -\pi \sim \pi$);

q, dimensionless heat flux, $Qa/c_p\mu(T_w - T_m)$;

q_w, dimensionless heat flux at the wall;

q_{wm}, mean value of q_w around the periphery ($\psi = -\pi \sim \pi$);

q_r, q_ψ, q_z, dimensionless heat flux in the fluid;

R, radius of curvature of the pipe axis;

Re, Reynolds number, $\equiv (2aW_m/\nu)$;

r, co-ordinate in radial direction in the cross section;

S, total area of wall in a spiral pipe;
T, temperature;
T_m, mixed mean fluid temperature;
T_{mL}, temperature giving h_m [°C];
T_w, wall temperature;
T_0, fluid temperature at the pipe inlet, or mixed mean temperature of cold fluid at the inlet or the outlet [°C];
T_1, mixed mean temperature of hot fluid at the inlet or the outlet [°C];
ΔT_{lm}, logarithmic mean temperature difference;
U, component of velocity in r-direction, $u = Ua/\nu$;
V, component of velocity in ψ-direction, $v = Va/\nu$;
W, component of velocity in z-direction, $w = Wa/\nu$;
W_m, mean velocity;
X, equation (29);
x, $\eta \cos \psi$;
y, $\eta \sin \psi$;
Z, co-ordinate along the pipe axis, $R\theta$, $z = Z/a$.

Greek symbols
α_i, eigenvalue;
α'_0, minimum eigenvalue;
δ, thickness of the boundary layer divided by the radius of pipe, a;
δ_T, thickness of the thermal boundary layer divided by the radius of pipe, a;
δ_m, mean value of δ around the periphery ($\psi = -\pi \sim \pi$);
ζ, δ_T/δ;
H, $\equiv R/a$;
η, $\equiv r/a$;
θ, axial co-ordinate;
μ, viscosity;
ν, kinematic viscosity;
ξ, $1 - \eta$;
ρ, density;
τ, wall temperature gradient in the case of uniform heat flux (constant);
ψ, co-ordinate in circumferential direction in the cross-section perpendicular to the pipe axis.

Subscripts
0, value at the pipe wall;
1, value at the flow core region;
c, a curved pipe;
s, a straight pipe;
δ, value at $\xi = \delta$;
δ_T, value at $\xi = \delta_T$.

1. INTRODUCTION

FORCED convective heat transfer in a curved pipe is found widely in such various industrial equipments as spiral tube heat exchangers. However, only a few papers have been reported about this problem and sufficient reliable data are not available. The studies made up so far by other investigators are mainly experimental ones. These experiments do not always interpret the cause of increase in pressure drop and heat-transfer rate due to curvature.

The purpose of the present study is to examine heat-transfer mechanism in a curved pipe theoretically and experimentally, and to present the design data which are important for practical use. From a practical point of view, the case when a secondary flow due to curvature gives a remarkable effect on heat-transfer coefficient is analysed in our research. The authors have already analysed theoretically the flow and temperature fields in fully developed laminar and turbulent flows in the first [1] and second [2] reports. The results of theoretical analyses have been ascertained by experiments. These theoretical and experimental studies were done under the condition of uniform heat flux. This wall temperature condition is often found in a counter-flow heat exchanger.

On the other hand, we have another wall temperature condition where fluid flowing through a spiral tube is heated or cooled under the condition of uniform wall temperature.

Therefore, for practical purposes, it is also necessary to examine heat transfer in a curved pipe for the case when the wall temperature is kept uniform. The former part of this report is devoted to theoretical analysis of temperature field in the region not close to the pipe inlet under the condition of uniform wall temperature.

In the previous papers we have discussed the velocity and temperature fields both of which are fully developed toward pipe axis. Usually a straight pipe is jointed to the inlet of the curved pipe and Ito [3] reported that the influence of curvature goes upstream in a straight pipe thirty times of a pipe diameter as far from the curved pipe inlet. This fact may be understood as to show that a secondary flow begins to develop fairly upstream before the inlet of a curved pipe. Moreover, the secondary flow tends to accelerate development of the flow and temperature fields by its mixing effect. This extremely shortens the entrance length of a flow in a curved pipe which is defined as the length from the curved pipe inlet to the beginning of the fully developed region. Therefore, in the case of a pipe coiled several times, the influence of the entrance region on flow resistance and heat transfer is negligibly small. In such cases, the resistance coefficients and the Nusselt number formulae obtained in the previous papers [1, 2] can be used along almost a whole length of a coiled pipe. However, when the coiled pipe is very long, there are cases when the fluid temperature difference between the pipe inlet and the outlet is very large, and the physical properties vary considerably. In these cases, if the temperature to be used for evaluation of physical properties in the formulae is known, the formulae for heat-transfer calculation become useful for design engineers.

For this purpose cases are studied when Prandtl number is near unity or more, and variation of temperature in a cross section perpendicular to a pipe axis is not so large and physical properties in a cross section are regarded as constant. The Nusselt number formulae obtained so far are arranged to be in simple forms so that they are referred conveniently. By the direct use of these simpler formulae, the temperature giving physical properties for the calculation of average Nusselt number of a long coiled pipe is shown for air, water and oil.

2. THEORETIC ANALYSIS OF TEMPERATURE FIELD UNDER THE CONDITION OF UNIFORM WALL TEMPERATURE

As shown in Fig. 1, when a fluid flows through a pipe with a uniform wall temperature, the temperature profile is changed from the uniform profile at the pipe inlet to the characteristic one at downstream. In the figure, Z is a co-ordinate along the pipe axis, R is a radius of curvature of pipe axis and θ is an angle of curvature, whence $Z = R\theta$. Co-ordinates in a cross section perpendicular to pipe axis are shown in Fig. 2, where r is a radial co-ordinate and ψ is a peripheral co-ordinate. Figure 1 shows the temperature profile in the plane $\overline{AA}$.

In order to analyse such a development of temperature distribution from the pipe inlet, the energy equation should be solved under the following boundary and initial conditions.

Boundary condition.

at $r = a$, $T = T_w$

where T is temperature and a is a radius of the pipe.

Initial condition:

at $Z = 0$, $T = T_0$.

The solution may be expressed in the following form:

$$T_w - T = \sum_{i=0}^{\infty} f_i \exp[-\alpha_i' Z] \tag{1}$$

where α_i' is an eigen value and $f_i(r, \psi)$ is an eigen function.

It may be difficult to obtain a solution for the whole region in the case when temperature distribution is distorted by such a secondary flow as that in a curved pipe. Let us consider the temperature distribution in the region not close

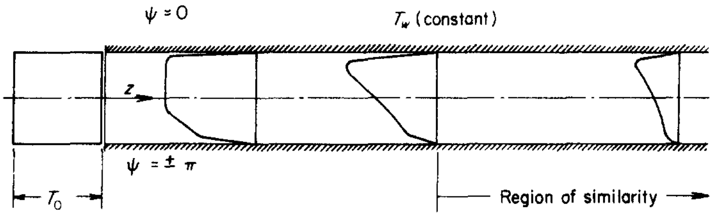

FIG. 1. Development of temperature distribution.

to the pipe inlet where Z is large enough. When we write the smallest eigen value as α_0', the term $f_0 \exp[-\alpha_0' Z]$ becomes predominant in equation (1). In that region, regardless of the

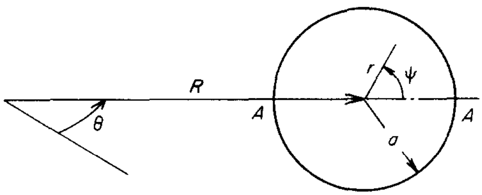

FIG. 2. System of coordinates.

initial condition, the temperature difference between the wall and fluid is written approximately from equation (1) as follows:

$$T_w - T = f_0 \exp[-\alpha_0' Z]. \tag{2}$$

The definition of a mixed mean temperature T_m is given as

$$T_m = \frac{1}{\pi a^2 W_m} \int_{-\pi}^{\pi} \int_0^a W T r \, \mathrm{d}r \, \mathrm{d}\psi \tag{3}$$

where W is a velocity component in the direction of $Z(\theta)$; and W_m is a mean velocity.

Substitution of equation (2) into equation (3) yields

$$T_w - T_m = \frac{\exp[-\alpha_0' Z]}{\pi a^2 W_m} \int_{-\pi}^{\pi} \int_0^a W f_0 r \, \mathrm{d}r \, \mathrm{d}\psi. \tag{4}$$

Define a dimensionless temperature g as

$$\frac{T_w - T}{T_w - T_m} = g. \tag{5}$$

When the velocity distribution is fully developed, it can be shown from equations (2) and (4) that g is a function of only x and y, and does not change with Z. This fact is understood as temperature distribution profiles become almost similar in the region far downstream from the inlet as shown in Fig. 1. We call this region as a region of similarity and the analysis is made on this region. As mentioned in the introduction, the entrance region of a flow field in a curved pipe is usually short and in the case of a long coiled pipe almost the whole length of the pipe is reasonably regarded as the region of similarity.

Although use of dimensionless temperature g expressed by equation (5) lessens the complexity of the problem, the analysis is not so easy as the case of uniform heat flux. In the present analysis, the second order solution by boundary-layer approximation as found in the previous papers [1, 2] is not calculated and only the largest terms necessary for the calculation of the first order solution are taken into account in the following analysis.

All quantities are put in the dimensionless forms as follows, where U and V are velocity components in the r, ψ-directions respectively; p, pressure; ρ, density; μ, viscosity; ν, kinematic viscosity; c_p, specific heat of fluid at constant pressure; k, heat conductivity of fluid, and Q is heat flux.

$$\eta = r/a; \quad H = R/a; \quad z = Z/a (= H\theta);$$
$$u = Ua/\nu; \quad v = Va/\nu; \quad w = Wa/\nu;$$
$$P = (a^2/\nu^2)(p/\rho); \quad C = -\partial P/\partial z.$$

Reynolds number, $Re = 2aW_m/\nu$;

$$\alpha_0 = a\alpha_0', \quad q = Qa/c_p\mu(T_w - T_m).$$

2.1 *Analysis for the Flow Field*

As pointed out in the previous papers [1, 2], in the range of practical importance, a flow field in a curved pipe differs greatly from that in a straight pipe. Namely, the centrifugal force causes a secondary flow, and stresses caused by the secondary flow predominate over the entire core region of the cross section of the pipe. A velocity distribution in the region except in a thin layer adjacent to the pipe wall has a relatively gentle gradient as shown in Fig. 3. On the other hand, in the thin region adjacent to the wall a velocity distribution has a steep gradient and this thin layer may be called a boundary layer. The secondary flow in the core region is expressed practically by a uniform flow toward the outer side of curvature. The dimensionless velocity of the uniform secondary flow in the core is denoted by D, and the dimensionless thickness of the boundary layer is denoted by δ. Since the variation of δ with peripheral angle is small, its variation is neglected and δ is replaced by its peripheral mean value δ_m.

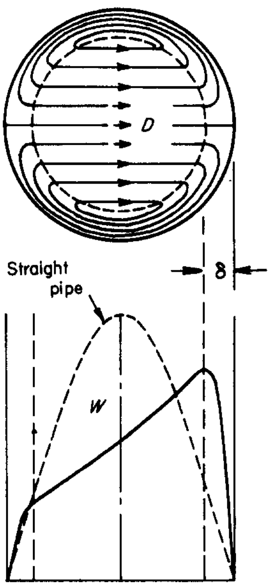

FIG. 3. Velocity distribution.

As in the previous papers [1, 2], the velocity components in the core region are expressed as follows:

$$\left.\begin{aligned} u_1 &= D\cos\psi \\ v_1 &= -D\sin\psi \\ w_1 &= A + \frac{C}{D}\eta\cos\psi \end{aligned}\right\} \quad (6)$$

where the suffix 1 denotes the values in the core and A = const. in both laminar and turbulent regions. The dimensionless pressure gradient C is expressed by the following equation when the $\frac{1}{7}$ power law is assumed for the turbulent velocity profile:

$$C = \begin{cases} \dfrac{2Re}{\delta_m} \quad \text{(laminar flow)} & (7) \\ 0{\cdot}0134\, Re^{\frac{7}{4}}\, \delta_m^{-\frac{1}{4}} \quad \text{(turbulent flow).} & (8) \end{cases}$$

Velocity distributions are expressed in terms of D and δ_m, and D and δ_m are obtained as follows:

$$D = \begin{cases} 0{\cdot}9656\, K^{\frac{1}{2}} \quad \text{(laminar flow)} & (9) \\ 0{\cdot}0852\, Re^{\frac{4}{5}} \left(\dfrac{a}{R}\right)^{\frac{1}{10}} \quad \text{(turbulent flow)} & (10) \end{cases}$$

$$\delta_m = \begin{cases} 4{\cdot}6311\, K^{-\frac{1}{2}} \quad \text{(laminar flow)} & (11) \\ 0{\cdot}2566 \left\{Re \left(\dfrac{a}{R}\right)^2\right\}^{-\frac{1}{5}} \quad \text{(turbulent flow)} & (12) \end{cases}$$

where K is so called Dean number

$$[= Re\sqrt{(a/R)}].$$

2.2 *Analysis of Temperature Field*

(a) *Fundamental equations*

By using non-dimensional quantities the energy equation is expressed as follows:

$$\frac{\partial}{\eta\,\partial\eta}(\eta q_\eta) + \frac{\partial q_\psi}{\eta\,\partial\psi} - \alpha_0 q_z = 0 \qquad (13)$$

where q_η, q_ψ and q_z are dimensionless heat flux in the η, ψ and z-directions respectively. As shown in the previous papers, heat fluxes are expressed as

$$\left.\begin{aligned} q_\eta &= -\frac{1}{Pr}\frac{\partial g}{\partial \eta} + ug \\ q_\psi &= -\frac{1}{Pr}\frac{1}{\eta}\frac{\partial g}{\partial \psi} + vg \\ q_z &= wg \end{aligned}\right\} \quad (14)$$

where Pr is Prandtl number, and heat flux in z-direction due to heat conduction is ignored. In the case of turbulent flow, u, v, w and g denote time mean values, and heat fluxes caused by turbulent fluctuation should be added to each right-hand side of equation (14). However, when deriving temperature distribution in the flow core region, the contribution by turbulent fluctuation can be neglected as well as the terms due to heat conduction.

(b) *Determination of* α_0

In order to calculate α_0 we use the energy equation expressing heat balance on a whole cross section of the pipe. In consideration of heat balance as shown in Fig. 4 and putting an

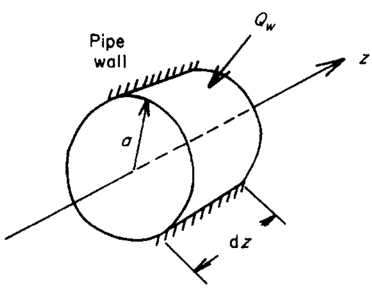

FIG. 4. Heat balance of fluid.

inward heat flux at the wall Q_w, we have the following relation:

$$\int_{-\pi}^{\pi} Q_w a \,\mathrm{d}\psi = \rho c_p \frac{\partial}{\partial Z} \int_{-\pi}^{\pi}\int_0^a W T r \,\mathrm{d}r\,\mathrm{d}\psi. \quad (15)$$

Equation (15) is transformed into the following non-dimensional form:

$$q_{wm} = \frac{\alpha_0}{4} Re \quad (16)$$

where q_{wm} is the mean value around the periphery ($\psi = -\pi \sim \pi$) of non-dimensional heat flux q_w.

(c) *Flow core region*

On the assumption that heat transfer by a secondary flow in equation (14) is predominant in the core region, heat fluxes are expressed as follows:

$$\left.\begin{aligned} q_\eta &= u_1 g_1 \\ q_\psi &= v_1 g_1 \\ q_z &= w_1 g_1 \end{aligned}\right\} \quad (17)$$

Substituting equations (6) and (17) into equation (13) and using perpendicular coordinates $x = \eta \cos \psi$ and $y = \eta \sin \psi$ for the sake of simplicity, we have

$$D\frac{\mathrm{d}g_1}{\mathrm{d}x} = \alpha_0 \left(A + \frac{C}{D}x\right) g_1. \quad (18)$$

The solution for equation (18) is obtained as

$$g_1 = N \exp\left\{\frac{\alpha_0}{D}\left(Ax + \frac{C}{2D}x^2\right)\right\} \quad (19)$$

where N is constant.

The profile of g_1 by equation (19) is like that shown in Fig. 5. The figure shows a similar profile to that obtained under the condition of uniform heat flux. As we are discussing the temperature field in the core region, g_1 given by equation (19) may be expanded as:

$$\begin{aligned} g_1 = N &\left\{1 + \frac{\alpha_0}{D}\left(Ax + \frac{C}{2D}x^2\right) + \frac{\alpha_0^2}{2D^2}\left(Ax + \frac{C}{2D}x^2\right)^2 + \frac{\alpha_0^3}{6D^3}\left(Ax + \frac{C}{2D}x^2\right)^3 + \ldots\right\} \quad (20) \\ = N &\left\{1 + \frac{\alpha_0 A}{D}x + \frac{\alpha_0}{2D^2}(C + \alpha_0 A^2)x^2 + \frac{\alpha_0 A}{D}\left(\frac{\alpha_0 C}{2D^2} + \frac{\alpha_0^2 A^2}{6D^2}\right)x^3 + \ldots\right\}. \quad (21) \end{aligned}$$

The constant N is determined from equation (5) giving the definition of g so as to satisfy the relation:

$$\frac{2}{\pi \, Re} \int_{-\pi}^{\pi} \int_{0}^{1} wg\eta \, d\eta \, d\psi = 1. \tag{22}$$

The integration of equation (22) is done in consideration that the boundary layer is extremely thin, and g_1 and w_1 may be extended

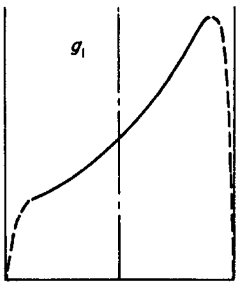

FIG. 5. Distribution of g_1.

over a whole cross section. By taking into account the terms to x^2 in the expansion of equation (21), the following relation is obtained

$$N = \frac{1}{1 + \frac{3}{8}\frac{\alpha_0 C}{D^2} + \frac{1}{8}\frac{\alpha_0^2 A^2}{D^2}}. \tag{23}$$

(d) *Boundary layer*

(I) *Laminar flow.* In case of a laminar flow, q_w is given by the temperature gradient in radial (η) direction at the wall. Dimensionless distance from the pipe wall is denoted by ξ ($= 1 - \eta$). In order to take into account the effect of Prandtl number, a thermal boundary layer of thickness δ_T is supposed along the pipe wall. The distribution of dimensionless temperature g in the boundary layer should be assumed to solve an energy integral equation according to $\delta_T \leqslant \delta$ and $\delta_T \geqslant \delta$.

In case of $\delta_T \leqslant \delta$, the boundary conditions are

at $\xi = 0, \quad g = 0$

$\xi = \delta, \quad g = g_{1\delta}.$

By taking into account the terms out of g in the first report [1] enough to satisfy these conditions, g is written that

$$g = g_{1\delta}\left\{\frac{2}{\zeta}\left(\frac{\xi}{\delta} - 2\frac{\xi^2}{\delta^2} + \frac{\xi^3}{\delta^3}\right) + 3\frac{\xi^2}{\delta^2} - 2\frac{\xi^3}{\delta^3}\right\} \tag{24}$$

where $g_{1\delta}$ denotes g_1 at $\xi = \delta$, ζ is the ratio δ_T/δ and a function of Pr as obtained later.

In case of $\delta_T \geqslant \delta$, g_1 as shown in equation (21) is applicable to the edge of the temperature boundary layer, and g is determined so as to satisfy the following boundary conditions [1]:

at $\xi = 0, \quad g = 0$

$\xi = \delta_T, \quad g = g_{1\delta_T}$

where $g_{1\delta_T}$ denotes g_1 at $\xi = \delta_T$.

$$g = g_{1\delta_T}\left(2\frac{\xi}{\delta_T} - \frac{\xi^2}{\delta_T^2}\right). \tag{25}$$

Dimensionless heat flux at the wall q_w is expressed as

$$q_w = \frac{1}{Pr}\left(\frac{\partial g}{\partial \xi}\right)_0 \tag{26}$$

where the suffix 0 denotes the value at the wall.

Equation (24) or (25) is substituted into equation (26). In equations (24) and (25) $g_{1\delta}$ and $g_{1\delta_T}$ are obtained by putting in equation (21) $x = (1 - \delta) \cos \psi$ and $(1 - \delta_T) \cos \psi$ respectively. Since δ and δ_T are very small compared with unity, we may put $x \approx$ cost ψ and find

$$g_{1\delta} \approx g_{1\delta_T}.$$

Then in both cases of $\delta_T \gtrless \delta$, equation (26) becomes

$$\begin{aligned} q_w &= q_{1\delta}\frac{2}{\zeta \, \delta \, Pr} \\ &= \frac{2N}{\zeta \, \delta \, Pr}\left[1 + \frac{\alpha_0}{2D^2}(C + \alpha_0 A^2)\cos^2\psi \right. \\ &\quad \left. + \frac{\alpha_0 A}{D}\left\{1 + \left(\frac{\alpha_0 C}{2D^2} + \frac{\alpha_0^2 A^2}{6D^2}\right)\cos^2\psi\right\}\cos\psi\right]. \end{aligned} \tag{27}$$

The mean value of q_w is obtained from equation (27). By using N in equation (23), we have

$$q_{wm} = \frac{2}{\zeta\, \delta_m\, Pr} X \tag{28}$$

where

$$X = \frac{1 + \dfrac{\alpha_0 C}{4D^2} + \dfrac{\alpha_0^2 A^2}{4D^2}}{1 + \dfrac{3\alpha_0 C}{8D^2} + \dfrac{\alpha_0^2 A^2}{8D^2}}. \tag{29}$$

For calculation of only the first approximation, δ is replaced by δ_m.

It is evident from equation (29) that X is almost equal to unity when $\alpha_0 C/D^2$ and $\alpha_0^2 A^2/D^2$ are small. The value of X from equation (29) is calculated and is shown in Fig. 6. The figure

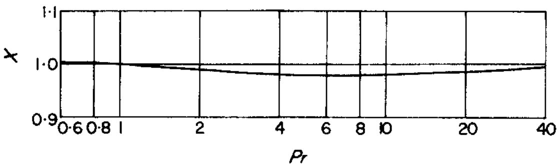

FIG. 6. X.

shows that X is reasonably assumed to be unity in the first approximation.

According this approximation, from equations (16) and (28) α_0 is given by

$$\alpha_0 = \frac{8}{\zeta\, \delta_m\, Re\, Pr}. \tag{30}$$

The ratio of temperature and velocity boundary-layer thickness ζ is obtained by considering heat balance in the boundary layer. We may put $\delta \ll 1$, $\eta = 1 - \xi \approx 1$, $\partial/\partial\eta = -\partial/\partial\xi \sim 0(\delta^{-1})$, $u \sim 0(D)$, $v \sim (D/\delta)$ and $\psi \sim 0(1)$.

From equations (13) and (14), the following boundary-layer equation is written

$$\frac{1}{Pr}\frac{\partial^2 g}{\partial \xi^2} + u\frac{\partial g}{\partial \xi} - v\frac{\partial g}{\partial \psi} + \alpha_0 w g = 0. \tag{31}$$

Integration of equation (31) from $\xi = 0$ to $\xi = \delta$ and use of equation of continuity yield

$$q_w = q_{1\delta}\frac{\partial}{\partial\psi}\int_0^\delta v\, d\xi - \frac{\partial}{\partial\psi}\int_0^\delta g v\, d\xi + \alpha_0\int_0^\delta w g\, d\xi \tag{32}$$

where $(1/Pr)(\partial g/\partial\xi)_0$ is written as q_w.

In case of $\delta_T \geqslant \delta$, the integration has to be done from $\xi = 0$ to $\xi = \delta_T$, however, since the difference between δ and δ_T is small, equation (32) is useful for both cases of $\delta_T \gtrless \delta$. The distributions of velocity components, v and w, in the boundary layer are written as explained in the first report [1].

$$v = \frac{12D}{\delta}\left(\frac{\xi}{\delta} - 2\frac{\xi^2}{\delta^2} + \frac{\xi^3}{\delta^3}\right)\sin\psi \tag{33}$$

$$w = \left(A + \frac{C}{D}\cos\psi\right)\left(2\frac{\xi}{\delta} - \frac{\xi^2}{\delta^2}\right). \tag{34}$$

The order of magnitude of the terms in right hand side of equation (32) is examined by using equations (33) and (34) in advance. The first and second terms have the order of magnitude $K^{\frac{1}{2}}$ due to D, where K is Dean number $[= Re\sqrt{(a/R)}]$. The third term has the order of magnitude $\alpha_0 A \delta_m$. This is written as the order of magnitude K^0 as $A = Re/2$. The present analysis is concerned with the range of large K, so that the third term is small in comparison with the other terms which are convective terms due to a secondary flow.

(i) The case of $\delta_T \leqslant \delta$. Equations (24) and (33) are substituted into the right-hand side of equation (32). The dimensionless heat flux is written in the following form:

$$q_w = E + F\cos\psi \tag{35}$$

where

$$E = \frac{\alpha_0 A}{2} N\left(1 + \frac{\alpha_0 C}{4D^2} + \frac{\alpha_0^2 A^2}{12D^2}\right) \tag{36}$$

$$F = \frac{D}{35}\left(22 - \frac{8}{\zeta}\right) N\left(1 + \frac{\alpha_0 C}{4D^2} + \frac{\alpha_0^2 A^2}{4D^2}\right). \tag{37}$$

In equation (36), E is replaced by its mean value about ψ [1].

The derivation of equations (36) and (37) is based on the linearization of $q_{1\delta}$ as follows:

$$g_{1\delta} = N\left\{1 + \frac{\alpha_0 A}{D}(1-\delta)\cos\psi\right.$$
$$+ \frac{\alpha_0}{2D^2}(C + \alpha_0 A^2)(1-\delta)^2\cos^2\psi$$
$$\left.+ \frac{\alpha_0 A}{D}\left(\frac{\alpha_0 C}{2D^2} + \frac{\alpha_0^2 A^2}{6D^2}\right)(1-\delta)^3\cos^3\psi + \ldots\right\} \tag{38}$$

$$\approx N\left\{1 + \frac{\alpha_0 C}{4D^2} + \frac{\alpha_0^2 A^2}{4D^2}\right.$$
$$\left.+ \frac{\alpha_0 A}{D}\left(1 + \frac{\alpha_0 C}{4D^2} + \frac{\alpha_0^2 A^2}{12D^2}\right)\cos\psi\right\}. \tag{39}$$

Using N in equation (23) and assuming X in equation (29) to be unity, we have from equation (37)

$$F = \frac{D}{35}\left(22 - \frac{8}{\zeta}\right). \tag{40}$$

The similar procedure can be employed about E in equation (36). Namely, as $\alpha_0 C/D^2$ is smaller than unity and $\alpha_0 C/D^2 \gg \alpha_0^2 A^2/D^2$, we approximately have the following relation:

$$E = \frac{\alpha_0 A}{2} = \frac{\alpha_0 Re}{4}. \tag{41}$$

Equation (41) shows that the mean value of q_w around the periphery ($\psi = -\pi \sim \pi$) given by the energy balance equation in the boundary layer also satisfies equation (16).

From equations (35). (40) and (41), q_w is written that

$$q_w = \frac{\alpha_0 Re}{4} + \frac{D}{35}\left(22 - \frac{8}{\zeta}\right)\cos\psi. \tag{42}$$

On the other hand, q_w is also expressed by $(\partial g/\partial\xi)_0$. We have from equations (24) and (26)

$$q_w = \frac{\alpha_0 Re}{4} + \frac{\alpha_0 Re}{\zeta D\delta_m Pr}\cos\psi. \tag{43}$$

By equating equations (42) and (43). we have the following equation for ζ

$$\frac{22}{35}\zeta^2 - \frac{8}{35}\zeta - \frac{8}{D^2\delta_m^2 Pr^2} = 0. \tag{44}$$

By substitution of D and δ_m of equations (9) and (11) into equation (44), ζ is determined as

$$\zeta = \frac{2}{11}\left\{1 + \sqrt{\left(1 + \frac{77}{4}\frac{1}{Pr^2}\right)}\right\} \tag{45}$$

Since $\zeta \leqslant 1$, $Pr \geqslant 1$.

(ii) The case of $\delta_T \geqslant \delta$. By substituting equations (25) and (33) into equation (32), we have the following equation from equation (32) just like the previous case

$$q_w = \frac{\alpha_0 Re}{4} + F'\cos\psi \tag{46}$$

where

$$F' = \frac{D}{5}\left(5 - \frac{4}{\zeta} + \frac{1}{\zeta^2}\right). \tag{47}$$

From equations (43) and (46), ζ is determined as

$$\zeta = \frac{1}{5}\left\{2 + \sqrt{\left(\frac{10}{Pr^2} - 1\right)}\right\} \tag{48}$$

since $\zeta \geqslant 1$, $Pr \leqslant 1$.

The results for ζ given by equations (45) and (48) are just the same as those in the first report [1].

(II) *Turbulent flow*. In case of turbulent flow, following the procedure shown in the second report [2], q_w can be reduced from Nusselt number formula of a straight pipe under the condition of uniform wall temperature, that is, q_w is determined according to the local law of turbulent heat transfer in a pipe, which can be derived from Nusselt number formula for a flow in a straight pipe by use of some appropriate assumptions [2]. It may be assumed with sufficient accuracy that the local law in a straight pipe is applicable to the flow near the wall of a curved pipe.

(*e*) *Nusselt numbers*

The definition of Nusselt number is given by

$$Nu = \frac{2aQ_{wm}}{k(T_w - T_m)} \tag{49}$$

where Q_{wm} is the mean value of Q_w around the periphery ($\psi = -\pi \sim \pi$) and T_m is defined by equation (3). For a curved pipe, use of the dimensionless quantities yields

$$Nu_c = 2\, q_{wm} Pr = \frac{\alpha_0}{2} Re\, Pr. \tag{50}$$

For laminar flow, by using α_0 in equation (30) and δ_m in equation (11), equation (50) is written that

$$Nu_c = \frac{0{\cdot}864}{\zeta} K^{\frac{1}{2}}. \tag{51}$$

It is found that equation (51) is quite the same as the Nusselt number formula in the first approximation which is obtained in the first report [1] under the condition of uniform heat flux. However, in the case of a straight pipe for uniform wall temperature,

$$Nu_s = 3{\cdot}66. \tag{52}$$

Then, Nu_c/Nu_s is different from the results of the first report [1] and it is written that

$$\frac{Nu_c}{Nu_s} = \frac{0{\cdot}236}{\zeta} K^{\frac{1}{2}}. \tag{53}$$

Table 1. Comparison of analytical procedure between two cases of wall temperature conditions

		Uniform heat flux	Uniform wall temperature
① g		$\dfrac{T_w - T}{\tau a}$	$\dfrac{T_w - T}{T_w - T_m}$
② $g_{1\delta_m}$		g_m	1
③ q_w		$\dfrac{Q_w a}{c_p \mu \tau a}$	$\dfrac{Q_w a}{c_p \mu (T_w - T_m)}$
④ q_{wm}		$\dfrac{Re}{4}$	$\dfrac{Re}{4}\alpha_0$
⑤ Nu_c		$\dfrac{2q_{wm} Pr}{g_m} = \dfrac{Re\, Pr}{2g_m}$	$2q_{wm} Pr = \dfrac{Re\, Pr}{2}\alpha_0$
⑥ q_{wm}	l	$\dfrac{2}{\zeta \delta_m Pr} g_{1\delta_m}$	$\dfrac{2}{\zeta \delta_m Pr} g_{1\delta_m}$
	t	$\hat{q}_w Pr^{-(1-\kappa)} \delta_m^{-\frac{1}{m}} [w_{1\delta}^{\frac{m-1}{m}}\, g_{1\delta}]_m$	
		$= L g_m \left(1 + \dfrac{M}{g_m}\right)$	$= L(1 + M\alpha_0)$
		where	
		$L = \hat{q}_w Pr^{-(1-\kappa)} \delta_m^{-\frac{1}{m}} A^{\frac{m-1}{m}}$,	$M = \dfrac{m-1}{m}\dfrac{C}{2D^2}$
		$\hat{q}_w = 2^{-\frac{m+1}{m}} \left\{\dfrac{(2m-1)^2}{m(4m-1)}\right\}^{\frac{m-1}{m}} \dfrac{4m-1}{2m+1}\beta$	
		$Nu_s = \beta Re^{\frac{m-1}{m}} Pr^{\kappa}$	

In case of turbulent flow, the analysis can be done in a similar way with that shown in the second report [2]. The result is shown in Table 1 in comparison with the result under the condition of uniform heat flux. In the first report [1], q_w is defined as $q_w = Q_w/k\tau$ where τ is a constant wall temperature gradient along the pipe axis. This definition is modified in Table 1.

The definition of non-dimensional temperature g is given by ① in Table 1 and the mean value at the edge of the boundary layer $g_{1\delta_m}$ becomes as shown in ② in the same Table.

For laminar flow, it is found by referring equations (40), (45) and (46) in the first report [1] that $g_{1\delta_m}$ is equal to g_m. For turbulent flow, although a slight difference is found by examination of the analysis in the second report [2], $g_{1\delta_m}$ is almost equal to g_m.

Dimensionless heat flux q_w is defined as ③ in the Table, and considering heat balance over a whole section, we have q_{wm}, mean value of q_w, as ④ in Table 1. Nusselt number is formed by q_{wm} and g_m (in the case of uniform wall temperature, $g_m = 1$). The analysis makes it clear that $1/g_m$ and α_0 are corresponding to each other.

Dimensionless heat flux q_w is expressed as ⑥ in Table 1. In case of turbulent flow, κ is constant, $m = 4$ or 5, and $\hat{q}_w$ is constant determined correspondingly to κ and m [2]. By equating ⑥ with ④ in Table 1, g_m and α_0 are obtained.

Table 1 clarifies that, in case of laminal flow, when ζ is the same, Nusselt numbers become equal for both cases of wall temperature condition.

It is reported by Seban and Shimazaki [4] that Nusselt number in a straight pipe for turbulent flow is hardly affected by the wall temperature condition. Therefore, $\hat{q}_w$ and κ in Table 1 are almost equal for both wall temperature conditions, and Nusselt number formulae for a curved pipe obtained in the second report [2] can be used for both cases.

The present analysis are limited in the first order solution of boundary-layer approximation, however correction by the second approximation can be considered small and may not be affected remarkably by the condition of wall temperature. Therefore, considering that Nusselt number formulae are the same for both cases, it may be concluded that Nusselt number formulae obtained under the condition of uniform heat flux can be applied with enough accuracy to the case where wall temperature is kept uniform.

3. PRACTICAL FORMULAE AND EVALUATION OF PHYSICAL PROPERTIES GIVING MEAN NUSSELT NUMBER

As pointed out in the introduction, the length of an entrance region is extremely short in a curved pipe compared with that in a straight pipe, and the influence of pipe inlet may be negligible in the case of a long spiral pipe. We restrict the study of the entrance region by showing the reported experimental results. It is reported by Ito [3] that pressure drop becomes proportional to pipe length when

$$\theta \sqrt{(R/a)} \geqslant 4{\cdot}7 \tag{54}$$

where θ is an axial angle shown in Fig. 2 measured from the inlet of a curved pipe. About heat transfer, experiments done by Seban and McLaughlin [5] show a similar result of the short entrance region.

Therefore, as mentioned in the introduction, the Nusselt number formulae obtained on the assumption that flow and temperature fields are fully developed can be used with enough accuracy for calculation of heat transfer rate in a long spiral pipe.

3.1. *Practical Formulae of Nu_c*

(a) *Laminar region*

In the first report [1], the formulae for Nusselt number obtained by the first and second order solutions of boundary-layer approximation are presented. They are shown by a straight and a curved line respectively in Fig. 7.

The analysis of resistance coefficient yields two lines similar to those shown in the figure, and the curve of the empirical formula given by Ito [3] enters between a straight and a curved line. The experimental results about heat transfer [1] prove that correlation equation may be expressed referring these two lines and the curve of the empirical resistance formula [3]. This correlated curve is shown by a dashed line in Fig. 7 and may be used for practical purposes.

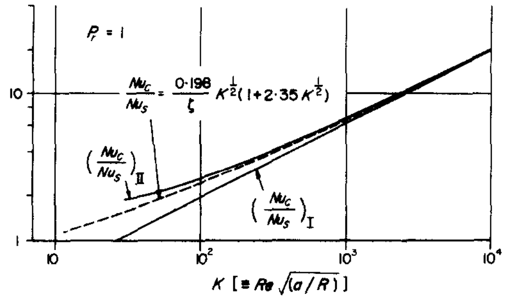

FIG. 7. $Nu_c/Nu_s - K$ diagram ($Pr = 1$).

The similar lines can be drawn for various values of Pr ranging from the neighbourhood of unity to infinity. They are approximately expressed by the following equation in a simpler form than the formulae in the previous paper [1].

$$Nu_c = \frac{0.864}{\zeta} K^{\frac{1}{2}}(1 + 2.35\, K^{-\frac{1}{2}}). \qquad (55)$$

The boundary-layer thickness ratio ζ is obtained from equations (45) and (48), or from Fig. 8. The applicable range of equation (55) is considered to be $k > 30(Pr = \infty) \sim 60(Pr \approx 1)$ by taking into account of availability of the theoretical analysis for resistance coefficient [1] and of experimental results of oil obtained by Seban and McLaughlin [5].

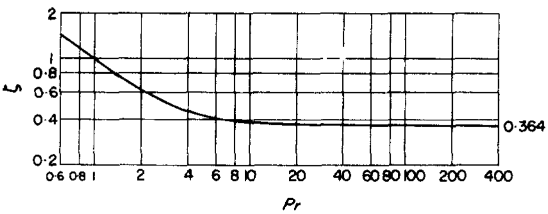

FIG. 8. ζ.

Critical Reynolds number is given by Ito [3].

$$Re_{cr} = 2 \times 10^4 \left(\frac{a}{R}\right)^{0.32}. \qquad (56)$$

(*b*) *Turbulent flow*

The results of the second report [2] are cited in the following.

For gases.

$$Nu_c = \frac{Pr}{26.2(Pr^{\frac{2}{3}} - 0.074)} Re^{\frac{4}{5}} \left(\frac{a}{R}\right)^{\frac{1}{10}} \times \left[1 + \frac{0.098}{\left\{Re\left(\frac{a}{R}\right)^2\right\}^{\frac{1}{5}}}\right]. \qquad (57)$$

This formula is applicable to the region, where $Pr \approx 1$ and $Re(a/R)^2 > 0.1$.

For liquids.

$$Nu_c Pr^{-0.4} = \frac{1}{41.0} Re^{\frac{5}{6}} \left(\frac{a}{R}\right)^{\frac{1}{12}} \times \left[1 + \frac{0.061}{\left\{Re\left(\frac{a}{R}\right)^{2.5}\right\}^{\frac{1}{6}}}\right]. \qquad (58)$$

Applicable range is

$$Pr > 1, \qquad Re(a/R)^{2.5} > 0.4.$$

In Fig. 9, theoretical curves of Nu_c are shown about the two values of R/a, 40 and 18.7, by taking Re as the abscissa. The curved pipes having the radius ratios of 40 and 18.7 were used in the authors' experiments [1] and [2], and the experimental results are also shown. The curves drawn on the left hand side in the figure are the analytical results obtained by the pertubation method [6]. They are calculated on the assumption that the effect of the secondary flow is very small. The results diverge rapidly with increasing Re.

It is clarified in Fig. 9 that the effect of R/a is less in turbulent region than in laminar region, while the gradient of Nu_c against Re is large for turbulent flow.

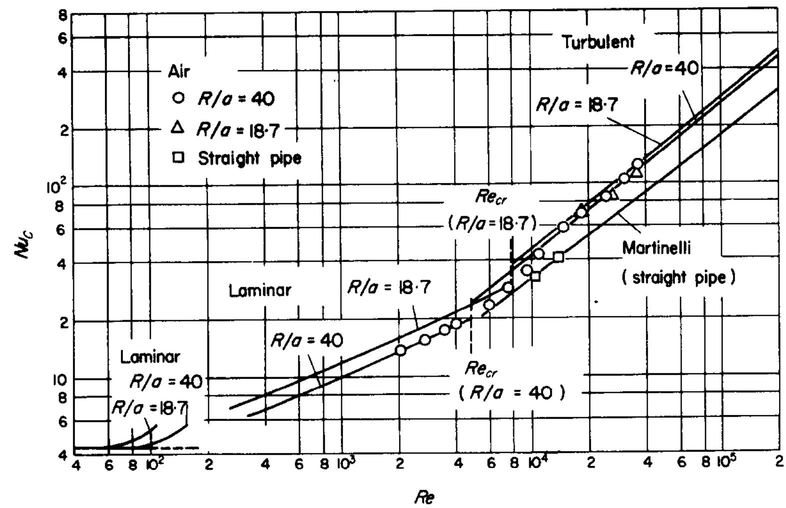

FIG. 9. Nusselt number in curved pipes of $R/a = 40$ and 18·7.

3.2. *Temperature giving Physical Properties*

In order to examine what physical properties should be used in calculating heat-transfer rate by means of Nusselt number formulae for a fully developed flow with a large temperature change, we express the mixed mean temperatures of fluid defined by equation (3) at the inlet and the outlet by T_0 and T_1 respectively as shown in Fig. 10. Physical properties change with mixed mean temperature T_m which increases or decreases from T_0 to T_1. In applications of spiral pipes as heat exchangers we often have a big temperature change, therefore, it is very important to consider the change of physical properties to calculate heat-transfer coefficient h given by

$$h = \frac{k}{2a} Nu_c. \tag{59}$$

Physical properties in the formulae for Nu_c and k change with temperature.

We introduce the following definition of a mean heat-transfer coefficient, when $T_1 > T_0$

$$h_m = \frac{1}{T_1 - T_0} \int_{T_0}^{T_1} h \, dT_m. \tag{60}$$

The total heat-transfer area over the complete length of a spiral pipe is expressed by S, and the net heat flux passing S is denoted by Q_T. Depending on the wall temperature condition, Q_T is written as follows:

Uniform heat flux:

$$Q_T = S(T_w - T_m)h_m \tag{61}$$

Uniform wall temperature:

$$Q_T = S\Delta T_{lm} h_m \tag{62}$$

where ΔT_{lm} is the logarithmic mean temperature difference given by

$$\Delta T_{lm} = \frac{T_1 - T_0}{\ln(T_w - T_0)/(T_w - T_1)}. \tag{63}$$

It seems convenient for practical use to find a particular temperature T_{mL} which gives h_m in equations (61) and (62). The procedure is described in the following. By substituting equation (59) into equation (60), and integrating in an appropriate temperature range from T_0 to T_1 chosen respectively for air, water and oil, we obtain h_m. The temperature which gives the same value of h with h_m is defined as T_{mL}. In equation (59), Nu_c is given from equation (55), (57) or (58) depending on whether a flow is laminar or turbulent.

Results are shown as follows, when T_0 is taken as a temperature at low-temperature side [°C] either inlet or outlet:

$$T_{mL} = T_0 + C_M(T_1 - T_0). \qquad (64)$$

Values of C_M are given in Table 2.

Temperature ranges taken in the calculations are:

air: $0°C \sim 200°C$

$1 < (T_1 + 273)/(T_0 + 273) < 2$

water: $10°C \sim 80°C$

oil: $20°C \sim 100°C$.

Table 2 shows that, when the Nusselt number formulae given by equations (55), (57) and (58) are used, physical properties at the arithmetic mean temperature between inlet and outlet may be taken in the calculations of heat-transfer coefficient for most cases.

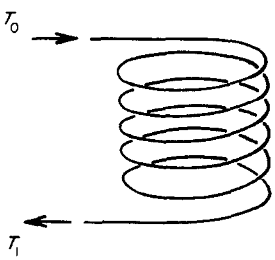

FIG. 10. Spiral pipe.

Table 2. The value of C_M

	Air	Water	Oil
Laminar	0·5	0·4	0·5
Turbulent	0·5	0·5	0·5

CONCLUSIONS

The study is made in addition to the previous reports [1] and [2] for the purpose of obtaining the practical method of evaluating heat-transfer rate in a spiral pipe and the following conclusive results are obtained.

(1) Theoretical analysis of the temperature field downstream from the pipe inlet under the condition of uniform wall temperature was made. The results prove that the formula of Nusselt number for uniform wall temperature case is the same with that for uniform heat flux case.

(2) For the convenience of calculating heat-transfer rate in a long spiral pipe, the analytical results for laminar flow in the previous paper [1] are arranged and written in a simpler form as a practical formula of Nusselt number. The practical formulae for turbulent flow [2] are also shown, and the applicable regions of every formula are subjected.

(3) Change of physical properties with temperature in these formulae is examined for the cases of air, water and oil. The results show that, except for the case of laminar flow of water, physical properties at the arithmetic mean temperature between inlet and outlet may be used.

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Résumé—Dans les articles précédents des auteurs, des études théoriques et expérimentales sur les champs de vitesses et de températures dans un tuyau courbe ont été faites dans le cas d'un flux de chaleur uniforme. Dans la première partie de l'article ci-dessous, on a analysé théoriquement le champ de températures loin à l'aval de l'entrée du tuyau dans le cas d'une température pariétale uniforme, en suivant le même processus que dans les articles précédents. On a trouvé que le nombre de Nusselt est modifié d'une façon remarquable par un écoulement secondaire dû à la courbure. Le résultat montre que dans l'approximation du premier ordre le nombre de Nusselt pour le transport de chaleur dans un tuyau courbe pour le cas de la température pariétale uniforme ne diffère pas de celui pour le cas du flux de chaleur uniforme, soit dans la région laminaire, soit dans la région turbulente.

Dans la dernière partie, les formules des nombres de Nusselt obtenues par les auteurs sont modifiées afin d'obtenir une expression plus simple pour un usage pratique.

On a aussi recherché quelle température devrait être choisie pour calculer les propriétés physiques lorsque ces formules de nombres de Nusselt sont employées.

Zusammenfassung—In den vorangegangenen Arbeiten der Autoren wurden die theoretischen und experimentellen Untersuchungen des Strömungs- und Temperaturfeldes in einem gekrümmten Rohr unter der Bedingung gleichmässigen Wärmeflusses durchgeführt. Im ersten Teil des gegenwärtigen Berichtes wird eine theoretische Analyse des Temperaturfeldes in grosser Entfernung stromabwärts vom Rohreinlauf für einheitliche Wandtemperatur gemacht, wobei wie in früheren Arbeiten verfahren wurde.

Es ergibt sich, dass die Nusselt-zahl durch die Sekundärströmung infolge der Krümmung erheblich beeinflusst wird. Das Ergebnis zeigt, dass Nusselt-zahlen für konstante Wandtemperatur oder konstanten Wärmefluss in erster Näherung nicht voneinander abweichen, sowohl für laminare als auch turbulente Bereiche.

Im zweiten Teil sind die für die Nusselt-zahlen gefundenen Gleichungen so umgestellt, dass sich zur praktischen Verwendung einfachere Ausdrücke ergeben.

Es wurde auch untersucht, welche Temperatur zur Bestimmung der Stoffwerte in diesen Nusselt-gleichungen zugrundegelegt werden soll.

Аннотация—В предыдущих работах автора теоретически и экспериментаьно изучались течение и температурные поля в изогнутой трубе в условиях однородного теплового потока. В предыдущей части данной работы теоретический анализ температурного поля на удалении от входа в трубу вниз по потоку в условиях изотермической стенки проводится по той же методике, что и в предыдущих работах. Найдено, что на число Нуссельта значительно влияют вторичные потоки, обусловленные кривизной стенки. Результаты в первом приближении указывают на отсутствие отличий чисел Нуссельта для искривленной трубы как в случае изотермической стенки, так и в случае однородного теплового потока и в ламинарной, и в турбулентной области.

Формулы для числа Нуссельта, полученные авторами, приведены к виду, достаточно простому для практического употребления.

Рассмотрен вопрос о выборе определяющей температуры при расчете физических свойств, входящих в расчетные формулы.